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Development of the flow field of a point force in an infinite fluid

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By means of similarity principles an analytical solution is constructed for the development of the linear flow field due to the instantaneous application of a constant point force in an infinite liquid. If the force is applied at the origin O and if r denotes distance from O, ν denotes the coefficient of kinematic viscosity of the fluid and t the time from the application of the force, the solution constructed exhibits the following features. Initially the flow field set up has a dipole structure with centre at O and axis along the direction of the impressed force. At a station r this dipole structure persists so long as $4\nu t \ll r^2$. In an axial cross-section the field lines form two sets of closed loops about two stagnation points in the equatorial plane. The stagnation points occur at $r = 1{\cdot}76(\nu t)^{\frac{1}{2}}$ and thus propagate to infinity with speed $0{\cdot}88(\nu/t)^{\frac{1}{2}}$. The steady state is reached algebraically.

1. Introduction

The steady-state flow field generated by point-force singularities has been studied for some time. The linear field or Stokeslet has been studied with various boundary conditions, especially recently, in connexion with various biological applications; for numerous references and applications in this field see Lighthill (1976). Lighthill (1978, §4.7) also suggested that in ultrasonics the attenuated energy of an acoustic beam generates a mean force that drives a velocity field, known as the 'sonic' wind, and thus when the acoustic power output of the beam is large the 'sonic' wind field resembles that of the classical Landau–Squire momentum jet.

There are not many solutions describing the transient state of these fields but recently Sozou & Pickering (1977) investigated the development of the flow field due to the instantaneous application of a constant point force F_0 at the origin O in an infinite fluid. They formulated the problem in terms of the dimensionless parameter $\lambda = (\nu t)^{\frac{1}{2}}/r$ and showed that 'initially', i.e. when $\lambda \ll 1$, the flow field has a dipole-like structure with centre at O and axis along the direction of the impressed force. As the parameter λ increases the nonlinear terms become important and the flow field develops, becoming asymmetric relative to the equatorial plane of the original dipole. In an axial cross-section the flow field consists of two sets of closed streamlines that propagate to infinity and when F_0 is increased the asymmetry and speed of eddy propagation are also increased. Eventually the flow field is transformed into that of the Landau–Squire momentum jet.

The solution constructed by Sozou & Pickering is a numerical solution, apart from the initial development stage. This initial state, together with the steady-state solution, suggests that the important limiting case of a very weak force, i.e. the linear model, possesses an analytic solution. This solution turns out to be very simple, yet it readily illustrates the various features of the development of the flow field, albeit without the asymmetry that characterizes the nonlinear regime, and gives a lower limit to the speed of eddy propagation to infinity. This solution, which complements the numerical solution of the nonlinear problem, is considered below.

2. Formulation and solution of the problem

We consider a viscous incompressible fluid occupying the whole space which is suddenly subjected to a constant force $\mathbf{F}_0$ applied at the origin O along the axis $\theta = 0$ of a spherical polar co-ordinate system (r, θ, ϕ). The flow field generated will obviously be symmetrical about the axis $\theta = 0, \pi$. We shall make use of a stream function ψ and note that its steady-state form, say ψ_s, is given by

$$\psi_s = Ar(1-\mu^2), \tag{1}$$

where $A = F_0/(8\pi\nu\rho)$, $\mu = \cos\theta$ and ρ is the fluid density. Also it was shown by Sozou & Pickering that the initial form of ψ, say ψ_i, which is valid when the parameter $\eta = r/[2(\nu t)^{\frac{1}{2}}]$ satisfies the condition $\eta \gg 1$, may be approximated by

$$\psi_i = Ar(1-\mu^2)/(2\eta^2). \tag{2}$$

Equations (1) and (2) suggest a possible solution of the form

$$\psi = Ar(1-\mu^2)\, g(\eta). \tag{3}$$

In terms of ψ the velocity field $\mathbf{v}$ is given by

$$\mathbf{v} = \left[\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, 0\right] \tag{4}$$

and in view of (3)

$$\mathbf{v} = Ar^{-1}[2\mu g, -(1-\mu^2)^{\frac{1}{2}}(g+\eta g'), 0]. \tag{5}$$

The linear momentum equation, except at the origin, is

$$\partial\mathbf{v}/\partial t = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{v}, \tag{6}$$

where p denotes the fluid pressure. Equations (1) and (6) [actually the curl of (6)] show that the solution of our problem can also be constructed by means of Laplace transforms, i.e. by setting $\psi = A(1-\mu^2)\,G(r,t)$. Here, however, we shall proceed with our similarity approach.

On dimensional grounds

$$p = A\nu\rho h(\mu, \eta)/r^2. \tag{7}$$

On taking the curl of (6) and making use of (5), after some algebra, we obtain

$$f'' = 2(\eta^{-1} - \eta^2)f', \tag{8}$$

where

$$f = \eta^2 g'' + 2\eta g' - 2g. \tag{9}$$

The fluid vorticity $\nabla\times\mathbf{v}$ is related to f by

$$\nabla\times\mathbf{v} = [0, 0, -A(1-\mu^2)^{\frac{1}{2}} r^{-2} f]. \tag{10}$$

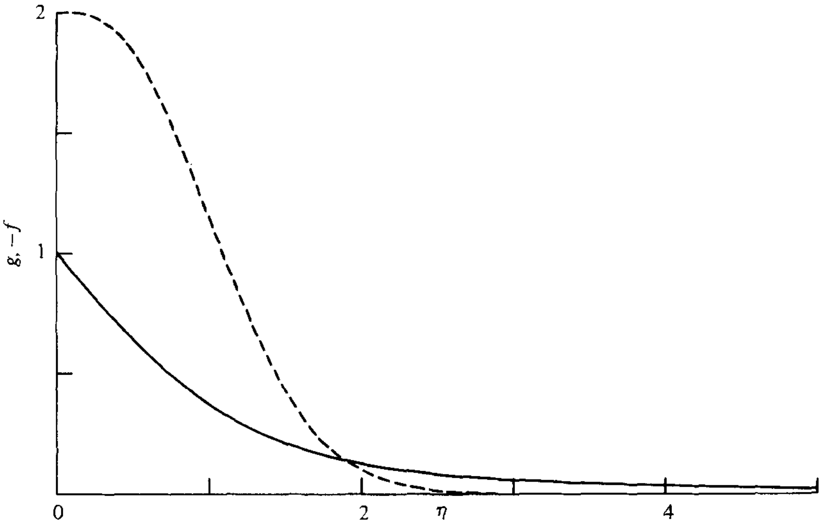

FIGURE 1. Values of g (continuous curve) and $-f$ (broken curve) as functions of η.

Initially, i.e. as $t \to 0$ and thus $\eta \to \infty$, the vorticity and velocity fields are zero and therefore

$$f(\infty) = 0, \quad g(\infty) = 0. \qquad (11), (12)$$

The steady-state solution ($t = \infty$, $\eta = 0$) requires that

$$g(0) = 1. \qquad (13)$$

The solutions of (8) and (9) that satisfy (11)–(13) are

$$f = -2 + \frac{8}{\pi^{\frac{1}{2}}} \int_0^\eta z^2 \exp(-z^2)\, dz = -2 + \frac{4}{\pi^{\frac{1}{2}}} \left(\int_0^\eta \exp(-z^2)\, dz - \eta \exp(-\eta^2) \right), \qquad (14)$$

$$g = 1 - \frac{2}{\pi^{\frac{1}{2}}} \int_0^\eta \exp(-z^2)\, dz + \frac{1}{\pi^{\frac{1}{2}}} \left(\int_0^\eta \exp(-z^2)\, dz - \eta \exp(-\eta^2) \right) \Big/ \eta^2. \qquad (15)$$

The function g was constructed by noting that η^{-2} is a solution for the complementary function of (9) and then setting $g = \eta^{-2} u(\eta)$. Some repeated integrals were performed by reversing the order of integration.

When $\eta \to \infty$, (14) implies that $f \simeq 0$ and (3) and (15) imply that

$$\psi = \frac{Ar}{2\eta^2}(1-\mu^2) = \frac{F_0}{4\pi\rho} \frac{t}{r} (1-\mu^2). \qquad (16)$$

Thus at the initial stage ($\eta \gg 1$) the flow field is irrotational and has a dipole-like structure with axis along the direction of the impressed force, as was shown by Sozou & Pickering. Our equation (16) is identical to equation (32) of the paper by Sozou & Pickering, where it was shown that in the nonlinear regime the dipole-like structure holds when $\eta \gg 1$ and $F_0 t^2 \ll 4\pi\rho r^4$.

It can easily be shown from (14) and (15) that as $t \to \infty$ and, therefore, $\eta \to 0$

$$f \simeq -2 + 8\eta^3/(3\pi^{\frac{1}{2}}), \quad g \simeq 1 - 4\eta/(3\pi^{\frac{1}{2}}) + 4\eta^3/(15\pi^{\frac{1}{2}}), \qquad (17), (18)$$

i.e. the steady state is reached algebraically.

It can also be shown, from (14) and (15), that the functions $-f(\eta)$ and $g(\eta)$ are monotonically decreasing functions of η. Details of the values of these functions are shown in figure 1. We note, however, that

$$\hat{\mathbf{e}}_{\phi}.\nabla\times\mathbf{v} = -\frac{A(1-\mu^2)^{\frac{1}{2}}}{4\nu t}\frac{f}{\eta^2}, \tag{19}$$

$$\psi = 2A(\nu t)^{\frac{1}{2}}(1-\mu^2)\,\eta g. \tag{20}$$

It can easily be shown from (14) and (19) that for a given t the maximum value of the vorticity occurs at $\eta = 0$, i.e. at the origin as might be expected. Also

$$\frac{d}{d\eta}(\eta g) = 1-\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\eta}\exp(-z^2)\,dz-\frac{1}{\pi^{\frac{1}{2}}}\left(\int_0^{\eta}\exp(-z^2)\,dz-\eta\exp(-\eta^2)\right)\Big/\eta^2. \tag{21}$$

The right-hand side of (21) is positive for $\eta \ll 1$ and negative for $\eta \gg 1$, i.e. ηg has a maximum, say at $\eta = \eta_0$. It then follows from (20) that, for a given t, ψ has a maximum at $\mu = 0$, $\eta = \eta_0$. The point $\mu = 0$, $r = 2\eta_0(\nu t)^{\frac{1}{2}}$ is a stagnation point and the streamlines form closed loops about it. The value of η that makes (21) zero is given by $\eta_0 \simeq 0{\cdot}88$. As the flow field develops this point propagates to infinity with a speed $dr/dt = \eta_0 \nu^{\frac{1}{2}}/t^{\frac{1}{2}}$. Streamlines of the flow field for various values of the parameter $T = (\nu t)^{\frac{1}{2}}/L$, where L is a characteristic length, are shown in figure 2. The general development of this flow field is similar to that of the nonlinear case, considered by Sozou & Pickering, but in the latter regime there is an asymmetry about the plane $\mu = 0$. It was shown, numerically, by these authors that if the maximum value of ψ occurs at (η_0, μ_0) then η_0 and μ_0 are larger the larger the applied force; that is, if F_0 is increased, the speed of eddy propagation to infinity and the flow-field asymmetry are also increased. The values $\eta_0 = 0{\cdot}88$ and $\mu_0 = 0$ that correspond to the linear case represent the lower limits of η_0 and μ_0 and correspond to the case $F_0 \rightarrow 0$. The corresponding values of (η_0, μ_0) for the cases $F_0 = 4{\cdot}61\nu^2\rho$, $34{\cdot}77\nu^2\rho$ and $156{\cdot}32\nu^2\rho$ are $(0{\cdot}90, 0{\cdot}05)$, $(0{\cdot}94, 0{\cdot}25)$ and $(1{\cdot}28, 0{\cdot}64)$.

On substituting (5) and (7) in (6), using (8), (9), (14) and (15), and equating the radial and transverse components on the two sides of the resulting equation, after some algebra we obtain

$$2h-\eta h_{\eta} = 4\mu, \quad h_{\mu} = 2. \tag{22), (23}$$

Hence $h = 2\mu$ and

$$p = F_0\mu/(4\pi r^2), \tag{24}$$

i.e. the pressure field is set up instantaneously. Equation (24) was also obtained by Sozou & Pickering as the initial pressure of the developing flow field. In that study, however, as t increased and the nonlinear terms became more significant p was modified until it reached its nonlinear steady-state form.

It can be shown, by considering the linear momentum of the fluid in a closed surface surrounding the origin, that the solution constructed above represents the flow field due to the application of a constant point force F_0 at O in the direction $\theta = 0$.

Since completing this work it has come to our notice that Happel & Brenner (1965, § 3.4) give expressions in Cartesian co-ordinates, derived by Burgers some 40 years ago, for working out the velocity field considered above. We note that, for the special

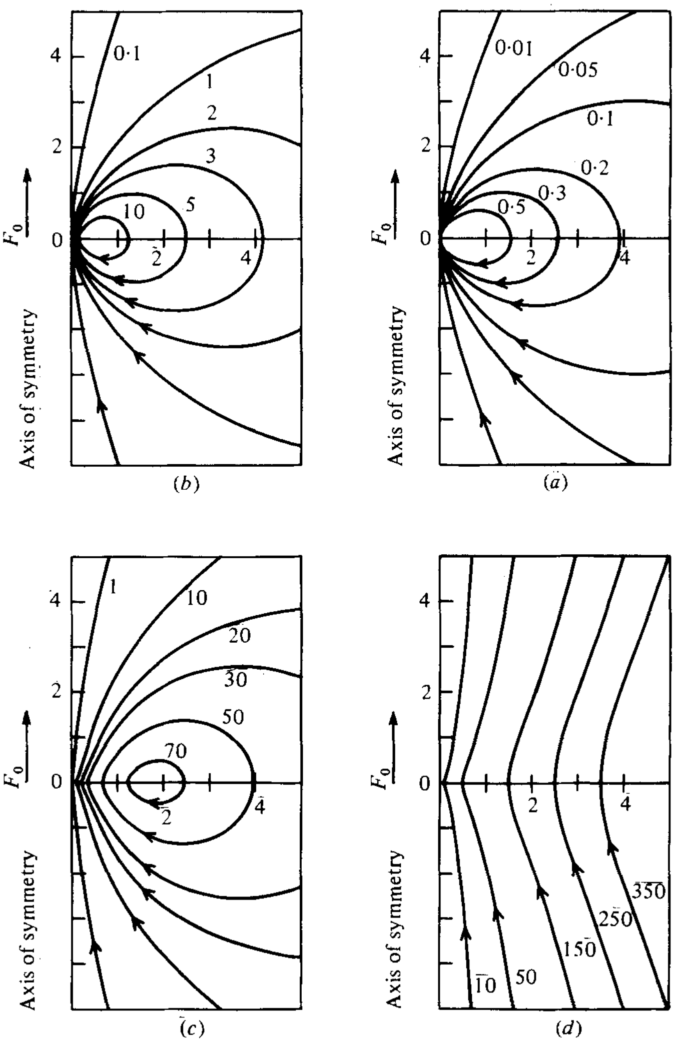

FIGURE 2. Streamlines of the developing flow field in a meridian plane for various values of the parameter $T = (\nu t)^{\frac{1}{2}}/L$, where L is a characteristic length. (*a*) $T = 0{\cdot}0625$, (*b*) $T = 0{\cdot}25$, (*c*) $T = 1$ and (*d*) $T = \infty$. The numbers on the curves are values of $800\pi\nu\rho\psi/F_0$. The distances along the axes are in units of L.

case considered here, the Burgers approach and expressions [Happel & Brenner 1965, equations (3-4.26) to (3-4.28) and (3-4.39)] are more complicated than the method and corresponding expressions presented above, and do not easily reveal the general structure of the flow field being set up. Happel & Brenner's ψ, which here we shall denote by Ψ in order to avoid confusion with our stream function, is given by their equation (3-4.39) as a double integral. We have transformed this into the single integral

$$\Psi = r\left[1-\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\eta}\exp\left(-\theta^2\right)d\theta+\frac{1}{\pi^{\frac{1}{2}}\eta^2}\left(2\eta-\eta\exp\left(-\eta^2\right)-\int_0^{\eta}\exp\left(-\theta^2\right)d\theta\right)\right]. \tag{25}$$

It can easily be shown that as $t \to \infty$ and, therefore, $\eta \to 0$ this expression reduces to

$$\Psi = r\left[1 - \frac{r}{3(\pi\nu t)^{\frac{1}{2}}} + \frac{r^3}{120\pi^{\frac{1}{2}}(\nu t)^{\frac{3}{2}}} - \dots\right]. \tag{26}$$

Also, as $t \to 0$ and, therefore, $\eta \to \infty$, apart from an additive term $4(\nu t)^{\frac{1}{2}}/\pi^{\frac{1}{2}}$, (25) may be approximated by

$$\Psi \simeq -2\nu t/r.$$

Thus the coefficient of the third term of Happel & Brenner's equation (3-4.40) and the sign of the right-hand side of their equation (3-4.41) are wrong. An indication of the accuracy of (26) is provided by working out for large t the axial velocity on $\mu = 1$. Equations (5) and (18) and also (26) [instead of their equation (3-4.40)] and their equation (3-4.28) with $F_x = F_y = 0$, $F_z = F_0$, $x = y = 0$ and $z = r$ give the same answer.

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